Mathematical aspects of gauge theory: lecture notes

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Some references are given at the end.

1 Basic Theory

Gauge theory=study of connections on fibre bundles

Let G be a Lie group. A principal G-bundle over a manifold M is a manifold P with a free right G action so that $P \to M = P/G$ is locally trivial, i.e. locally equivalent over $U \subset M$ to the trivial bundle $U \times G$ with the obvious action.

The associated bundle construction. Suppose G acts on the left on X. Then we can form $P \times_G X$, which is a bundle over M with fibre X.

The most important example is when G acts on a vector space (real or complex) V via a linear representation. Then we get a vector bundle over M.

Conversely, if $E \to M$ is (say) a rank r complex vector bundle we get a principal $GL(r, \mathbf{C})$ bundle of frames in E, and we recover E from the standard representation of $GL(r, \mathbf{C})$ on $\mathbf{C}^r$. If E has a Hermitian metric we get a principal $U(r)$ bundle of orthonormal frames.

In practice we can usually work interchangeably with principal bundles and vector bundles.

In a principal bundle P we have a vertical sub-bundle $V \subset TP$, the tangent space to the fibres. A **connection** on P is a sub-bundle $H \subset TP$ which is transverse to V (i.e. $TP = V \oplus H$) and invariant under the G-action. A connection can be thought of as an infinitesimal trivialisation of P.

Now we recall a fundamental notion in differential geometry. Let H be any sub-bundle of a manifold Q. We define a section $\mathcal{F}$ of $\Lambda^2 H^* \otimes (TQ/H)$ as follows. For each $q \in Q$ and $\xi_1, \xi_2 \in H_q$ we extend ξ_i locally to sections of H and set

$$\mathcal{F}_q(\xi_1, \xi_2) = [\xi_1, \xi_2] \bmod H,$$

where $[\ ,\ ]$ is the Lie bracket on vector fields. The formula

$$[f\xi_1, \xi_2] = f[\xi_1, \xi_2] + (\nabla_{\xi_2} f)\xi_1,$$

shows that this definition is independent of the choice of extensions. Then **Frobenius' Theorem** states that $\mathcal{F} = 0$ if and only if H is *integrable* (i.e. each point q has a neighbourhood which is a union of submanifolds with tangent space H).

Applying this to our situation, we get the *curvature* of a connection. In this case TP/H is identified with the vertical subspace V and hence with the Lie algebra $\mathrm{Lie}(G)$ via the derivative of the G action. Also H can be identified with $\pi^*(TM)$. So on the total space P the curvature is a section of the bundle $\Lambda^2(\pi^*(TM) \otimes \mathrm{Lie}(G)$. It is an exercise in the definitions to see that this gives a section F of the bundle $\Lambda^2 T^*M \otimes \mathrm{ad}P$ over M, where $\mathrm{ad}P$ is the vector bundle with fibre $\mathrm{Lie}(G)$ associated to the adjoint representation.

A more explicit description of a connection is given as follows. The right translation of G on itself defines the Maurer-Cartan form θ, a 1-form on G with values in $\mathrm{Lie}(G)$. This is invariant under G, acting by right translation on itself and by the (right) adjoint action on $\mathrm{Lie}(G)$. The structural equations for a Lie algebra can be expressed by the equation

$$d\theta + \frac{1}{2}[\theta, \theta] = 0.$$

Here d is the exterior derivative on 1-forms and $[\theta, \theta]$ combines the wedge product on 1-forms with the bracket on $\mathrm{Lie}(G)$. (Note that if G is a matrix group we can write $\theta = g^{-1}dg$.) Now given a connection on P we get a $\mathrm{Lie}(G)$ valued 1-form A on P from the projection to the vertical subspace (i.e. H is the kernel of A). This restricts to the Maurer-Cartan form on each fibre. One finds that $F = dA + \frac{1}{2}[A, A]$ is a "horizontal" 2-form and this gives the curvature $\mathcal{F}$. In a local trivialisation over $U \subset M$, by a section $\tau : U \to P$, we set $A^\tau = \tau^*(A)$ which is a $\mathrm{Lie}(G)$-valued 1-form on U. The curvature is given (in the corresponding local trivialisation of $\mathrm{ad}P$) by the same formula

$$F = dA^\tau + \frac{1}{2}[A^\tau, A^\tau].$$

Remark. If G is a matrix group, so the A^τ are matrices of 1-forms, we can write the quadratic term as $A^\tau \wedge A^\tau$, where we use matrix multiplication and the wedge product on forms.

An equivalent definition of a connection for most purposes goes through the notion of a covariant derivative. Let E be the vector bundle associated to P by some representation ρ. A connection A on P defines a covariant derivative

$$\nabla_A : \Gamma(E) \to \Gamma(T^*M \otimes E).$$

We can define this by saying that at a given point $x \in M$ we compute the covariant derivative of a section by taking the ordinary derivative in a trivialisation compatible with H at x. In general, in local co-ordinates x_i on M we have

$$\nabla_A s = \sum_i (\nabla_{A,i} s)\ dx_i$$

where

$$\nabla_{A,i} = \frac{\partial}{\partial x_i} + \rho(A_i^\tau).$$

The meaning here is that

- Sections of E are identified with vector-valued functions, in the trivialisation induced by τ.
- $A^\tau = \sum A_i^\tau dx_i$.
- We use the same symbol ρ for the induced representation of Lie algebras, so $\rho(A_i^\tau)$ are matrix valued functions which act on vector-valued functions in the obvious way.

From this point of view the curvature measured the failure of covariant derivatiues to commute:

$$\rho(F)_{ij} = [\nabla_{A,i}, \nabla_{A,j}] = \rho(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j]).$$

This is the same formula $\rho(F) = \rho(dA + \frac{1}{2}[A, A])$.

Remark. In the above we have dropped the trivialisation τ from the notation and in future we usually drop the representation ρ.

Two examples where these ideas enter.

1. **The falling cat**. A connection on $\pi : P \to M$ defines *parallel transport* along paths. If $\gamma : [0, 1] \to M$ and p is a point in $\pi^{-1}(\gamma(0))$ there is a unique horizontal lift to a path $\tilde{\gamma} : [0, 1] \to P$ with $\gamma = \pi \circ \tilde{\gamma}$, $\tilde{\gamma}(0) = p$ and the derivative of $\tilde{\gamma}$ taking values in H. If γ is a loop there is no reason why $\tilde{\gamma}$ should return to the initial point p. We can write $\tilde{\gamma}(1) = p.g$ for some $g \in G$. This is the *holonomy* of the connection.

   This gives a framework to discuss the process by which a falling cat is able to manoeuver itself to land on its feet. Assuming for simplicity that the cat cannot put itself into a symmetrical configuration, we have a principal $SO(3)$ bundle $\mathcal{C} \to \mathcal{C}_0$ where $\mathcal{C}$ is the space of configurations of the cat in $\mathbf{R}^3$ (with centre of mass 0) and $\mathcal{C}_0$ is the quotient by the rotations. While falling, the cat can choose a path γ in $\mathcal{C}_0$ and the laws of mechanics give a horizontal lift in $\mathcal{C}$. The cat has to choose a loop γ such that the holonomy gives the desired rotation (so that it lands feet down).

2. **Coupling to magnetic fields**. The "wave function" in quantum mechanics on $\mathbf{R}^3$ appears first as a complex valued function, but really it should be considered as a section of a complex line bundle, associated to a principal S^1-bundle. In electro-magnetism one considers a magnetic potential, a vector field A with $B = \mathrm{curl} A$. This magnetic potential is really a connection on the S^1 bundle (in a local trivialisation and identifying $\mathbf{R}^3$ with its dual in the usual way), and the magnetic field is the curvature. The prescription for replacing ordinary derivatives ∂_i by $\partial_i + A_i$ gives the coupling between the field and the wave function, for example in showing how to modify Schroedinger's equation in the presence of a magnetic field.

More theory

Any first order differential operator can be coupled to a connection. That is, if we have a first order linear differential operator $\mathcal{D} : \Gamma(U_0) \to \Gamma(U_1)$ (for vector bundles $U_i \to M$) and we have a connection A on a vector bundle E then we can define

$$\mathcal{D}_A : \Gamma(E \otimes U_0) \to \Gamma(E \otimes U_1).$$

In particular taking the exterior derivative on forms we get

$$d_A : \Omega^p(E) \to \Omega^{p+1}(E).$$

(Here $\Omega^p(E)$ is notation for $\Gamma(\Lambda^p T^* M \otimes E)$.)

Then we have

$$d_A^2 = F_A$$

where F_A acts algebraically by wedge product on forms and the derivative of the defining representation on E. This can be used as another definition of curvature.

The curvature satisfies the *Bianchi identity* $d_A F_A = 0$.

In this subject (gauge theory) there are three important classes of object.

- **Connections**. The set of all connections on a bundle P is an affine space modelled on the vector space $\Omega^1(\mathrm{ad}P)$. In terms of covariant derivatives $\nabla_{A+a} = \nabla_A + a$.
- **Curvature** $F(A) \in \Omega^2(\mathrm{ad}P)$. We have

$$F(A + a) = F(A) + d_A a + \frac{1}{2}[a, a].$$

- **Gauge transformations**. These are automorphisms of the bundle P covering the identity on M and they act on the connections. In terms of vector bundles,a gauge transformation is given by an everywhere invertible section of End$E$. The action is

$$\nabla_{g(A)} = g \circ \nabla_A \circ g^{-1} = \nabla_A - (\nabla_A g) g^{-1}.$$

The linearisation of the curvature map is given by

$$d_A : \Omega^1(\mathrm{ad}P) \to \Omega^2(\mathrm{ad}P),$$

and the linearised action of the gauge transformation is

$$-d_A : \Omega^0(\mathrm{ad}P) \to \Omega^1(\mathrm{ad}P).$$

Chern-Weil Theory

Let $\Phi : \mathrm{Lie}(G) \to \mathbf{R}$ be a homogeneous polynomial function of degree s which is invariant under the adjoint action of G on $\mathrm{Lie}(G)$. Then if A is a connection on a G-bundle $P \to M$ there is a well defined $2s$-form $\Phi(F_A)$ on M. The basic statement is that this is closed and the cohomology class of $\Phi(F_A)$ is independent of the choice of connection. This defines a *characteristic class* of P in $H^{2s}(M; \mathbf{R})$. The examples we shall use are:

- $G = U(r)$ and Φ is the $i/2\pi$ times the trace. This defines the first Chern class
$$c_1(P) = [\frac{i}{2\pi} \mathrm{Tr}(F_A)] \in H^2(M).$$
- $G = U(r)$ and
$$\Phi(\xi) = \frac{-1}{8\pi^2} (\mathrm{Tr}(\xi))^2 - \mathrm{Tr}(\xi^2).$$
This defines the second Chern class $c_2(P) \in H^4(M)$. In particular for $SU(r)$ bundles we have
$$c_2(P) = \frac{1}{8\pi^2}\ [\mathrm{Tr}(F_A^2)].$$

Flat connections.

A connection A with curvature 0 is called flat. The basic fact is that these correspond (up to equivalence) to conjugacy classes of representations $\pi_1(M) \to G$. In one direction, the representation is obtained from the holonomy of a flat connection. In the other direction, given a representation one finds a natural flat connection on the bundle $\tilde{M} \times_\pi G$, where $\tilde{M}$ is the universal cover of M and $\pi = \pi_1(M)$ acts on G via the representation and left multiplication in G.

MORE BACKGROUND

A connection is a differential-geometric concept, for example arising in the *Levi-Civita* connection of a Riemannian manifold. But the background for much of the material we discuss comes from two other areas: *Field Theory* in Mathematical Physics and *complex (algebraic) geometry*.

Field Theory

The standard model of particle physics involves a principle bundle P over space time M with structure group

$$G_{SM} = SU(3) \times SU(2) \times U(1).$$

The theory involves a connection on the bundle and various fields: sections of associated vector bundles. We will discuss the Riemannian version, so M is an oriented n-dimensional Riemannian manifold with an $SO(n)$ frame bundle $\Pi \to M$. Given representations ρ of G and σ of $SO(n)$ we get an associated vector bundle. The only aspect that we have time to go into in this course is that in addition it is crucial to use *spinor representations*.

Recall that for each n there is a double cover $\mathrm{Spin}(n) \to SO(n)$. A spin structure on M is a principal $\mathrm{Spin}(n)$ bundle $\tilde{\Pi} \to M$ which is a double cover of Π in the obvious sense. Now the vector bundles in the standard model arise from representations of G and $\mathrm{Spin}(n)$. The main point is that there is a representation of $\mathrm{Spin}(n)$ on a complex vector space S which does not factor through $SO(n)$, so on a spin manifold we have a vector bundle which we write as $S \to M$ and for any bundle E associated to a representation of G we have E-valued spinor fields, sections of $E \otimes S$.

The spin representation comes with a map of Spin-representations—Clifford multiplication:

$$c : S \otimes \mathbf{R}^n \to S.$$

Given a vector bundle E with connection we have a covariant derivative

$$\nabla_A : \Gamma(E \otimes S) \to \Gamma(E \otimes S \otimes T^*M).$$

(This uses also the connection on S induced by the Levi-Civita connection.) Composing with Clifford multiplication gives the *Dirac operator*

$$D_A : \Gamma(E \otimes S) \to \Gamma(E \otimes S).$$

Here is an inductive recipe for writing down the Dirac operator explicitly: we just work over $\mathbf{R}^n$ with the trivial bundle E, so spinor fields are S valued functions and we write temporarily S_n to indicate the space dimension.

If $n = 2m$ is even then S_{2m} splits as $S^+_{2m} \oplus S^-_{2m}$ and the Dirac operator D_{2m} is the sum of

$$D_+ : \Gamma(S^+_{2m}) \to \Gamma(S^-_{2m})$$

and its adjoint

$$D_- = D_+^* : \Gamma(S^-_{2m}) \to \Gamma(S^+_{2m}).$$

To go to the next dimension $2m+1$ we set $S_{2m+1} = S^{2m}$ and

$$D_{2m+1} = i\frac{\partial}{\partial t} + D_{2m}.$$

Here we are thinking of $\mathbf{R}^{2m+1} = \mathbf{R}^{2m} \times \mathbf{R}$ and t is the co-ordinate in the $\mathbf{R}$ factor.

To go from dimension $2m+1$ to $2m+2$ we take $S^+_{2m+2} = S^-_{2m+2} = S_{2m+1}$ and let

$$D^{\pm}_{2m+2} = \pm\frac{\partial}{\partial s} + D_{2m+1}$$

where now s denotes the last coordinate. The process can start in dimension 1, where the Dirac operator is $i\frac{d}{dx}$. We will come back to the Dirac operator in dimensions $2, 3, 4$ later.

Complex (algebraic) geometry

Now suppose that M is a complex manifold. The forms decompose into bi-type $\Omega^{p,q}$ and the exterior derivative as $d = \partial + \overline{\partial}$.

Let $E \to M$ be a holomorphic vector bundle (i.e. the total space is a complex manifold and the projection is holomorphic—-one can also work with holomorphic principal bundles with structure group a complex Lie group.)

There is an intrinsic differential operator

$$\overline{\partial}_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E).$$

A way to define this is to say that it is given by the ordinary $\overline{\partial}$ operator on vector-valued functions in a local holomorphic trivialisation, then see that this is independent of the trivialisation. This operator satisfies $\overline{\partial}_E^2 = 0$.

Now let A be a connection on E. We can write $d_A = \partial_A + \overline{\partial}_A$ and we say that A is compatible with the holomorphic structure if $\overline{\partial}_A = \overline{\partial}_E$. More geometrically this says that the horizontal subspaces in the frame bundle of E are complex linear subspaces.

A basic (easy) Lemma asserts that given a Hermitian metric on a holomorphic bundle E there is a unique connection compatible with both structures. In a holomorphic trivialisation the metric is given by a matrix-valued function h and the covariant derivative is

$$\nabla_A = \overline{\partial} + \partial + h^{-1}\partial h.$$

The curvature is $\overline{\partial}(h^{-1}\partial h)$.

A relatively difficult result is the *integrability theorem* which asserts that if we have a C^∞ vector bundle E with a connection A such that the curvature has no (0,2) component, $F^{0,2} = 0$, then there is an induced holomorphic structure on E with $\overline{\partial}_E = \overline{\partial}_A$. We can define a sheaf by local solutions of the equation $\overline{\partial}_A s = 0$—the point of the theorem is that if $F^{0,2} = 0$ there are "enough" solutions of this equation.

2 Dimension 2

Let Σ be a compact Riemann surface. We begin with the classical case of line bundles. Let L be a holomorphic line bundle over Σ. A hermitian metric h_0 on L defines a unitary connection with curvature F_{h_0}. This is an ordinary 2-form on Σ and the degree of the line bundle (i.e. the first Chern class evaluated on Σ) is

$$d = \frac{i}{2\pi} \int_\Sigma F_{h_0}.$$

Changing the metric to $h = e^\phi h_0$ changes the curvature by

$$F_h = F_{h_0} + \overline{\partial}\partial\phi.$$

The fundamental analytical result about compact Riemann surfaces is that any 2-form of integral 0 can be written as $\overline{\partial}\partial f$ for a function f. So any form representing c_1 can be realised as F_h for a suitable h. In particular if $d = 0$ there is a metric (unique up to constant factor) which endows L with a *flat connection* (i.e. curvature 0). In this way we get two descriptions of the Jacobian torus J.

- J parametrises isomorphism classes of holomorphic line bundles of degree 0. As such it is seen as a complex torus

$$J = H^{0,1}/H^1(\Sigma; \mathbf{Z}).$$

- J parametrises isomorphism classes of flat S^1-connections. As such it is seen as the real torus

$$J = \mathrm{Hom}(\pi_1, S^1) = H^1(\Sigma; \mathbf{R})/H^1(\Sigma; \mathbf{Z}).$$

Another way of expressing this is through the "Hodge" isomorphism

$$H^{0,1} = H^1(\Sigma; \mathbf{R}).$$

(as real vector spaces).

In the second description we see a natural symplectic structure on J, induced by the cup product on $H^1(\Sigma; \mathbf{R})$.

Now consider the higher rank case, of a rank r holomorphic bundle $E \to \Sigma$. A hermitian metric h gives a curvature F_h and the degree is the integral of $(i/2\pi)\mathrm{Tr}(F_h)$. The theorem of Narasimhan and Seshadri asserts that a bundle of degree 0 has a metric with $F_h = 0$ if and only if it is *polystable*.

We recall the definition (in the paragraphs below we write "bundle" for "holomorphic bundle"). For any bundle V the *slope* of V is

$$\mathrm{slope}(V) = \frac{\mathrm{degree}(V)}{\mathrm{rank}(V)}.$$

A bundle V is *stable* if for all proper sub-bundles $V' \subset V$ we have

$$\mathrm{slope}(V') < \mathrm{slope}(V).$$

(It is equivalent to say that for all proper quotients V'' we have $\mathrm{slope}(V) < \mathrm{slope}(V'')$.)

A bundle is *polystable* if it is either stable or a sum of stable bundles all with same slope.

- A slight variant of the Narasimhan-Seshadri statement is to say that a degree 0 bundle admits an irreducible flat connection (obvious definition) if and only if it is stable.
- There is a more general statement to cover bundles of non-zero degree which we will come to shortly.

The Narasimhan-Seshadri theorem gives an algebro-geometric criterion for the existence of a differential geometric object. But the condition (stability) arises naturally within algebraic geometry when considering *moduli* questions. For a given rank and degree we can define the moduli set of bundles up to isomorphism but this will not usually have a good structure. We see this we review the theory of *extensions*.

Given bundles V', V'' the isomorphism classes of extensions

$$0 \to V' \to V \to V'' \to 0$$

are classified by the cohomology group $H^1(\mathrm{Hom}(V'', V'))$. Multiplying the cohomology class by a non-zero factor gives the same bundle V. The class 0 in H^1 corresponds to the direct sum $V' \oplus V''$. So if B is a nonzero element in H^1, giving a bundle V, the bundles V_t determined by tB are isomorphic to V for $t \neq 0$ but to the direct sum when $t = 0$, which will typically not be isomorphic to V. This shows that the moduli set of all isomorphism classes of bundles is not Hausdorff in its natural topology. The point of the definition of (poly)stability is that if we restrict to this class we do get a good moduli space.

The correspondence between extension and H^1 classes can be obtained differential geometrically as follows. Given an extension, we choose a Hermitian metric on V which gives a C^∞ isomorphism $V = V' \oplus V''$. The metric on V induces metrics on V', V'' so we have connections on all three bundles. Now we can write

$$\nabla_V = \begin{pmatrix} \nabla_{V'} & \beta \\ -\beta^* & \nabla_{V''} \end{pmatrix},$$

where β is a 1-form with values in $\mathrm{Hom}(V'', V')$. The fact that V' is a holomorphic sub-bundle means that $\beta \in \Omega^{0,1}(\mathrm{Hom}(V'', V'))$ and β gives a representative of the extension class in Dolbeault cohomology.

We can use this theory to show that a bundle which admits a flat unitary connection is polystable. The "diagonal" components of the curvature of the connection on V take the form

$$F_V = \begin{pmatrix} F_{V'} - \beta\beta^* & * \\ & F_{V''} - \beta^*\beta \end{pmatrix}.$$

One checks that $i\ \mathrm{Tr}(\beta\beta^*) = -|\beta|^2$ (if we choose a compatible metric on Σ, to identify 2-forms with functions). So if $F_V = 0$ then $i\mathrm{Tr}F(V') \leq 0$ and Chern-Weil implies that the degree of V' is ≤ 0. Moreover equality holds if and only if $\beta = 0$ and then $V = V' \oplus V''$ and V', V'' have flat unitary connections. The statement follows by induction on rank.

We will now sketch a proof of the (harder) converse: a polystable bundle of degree 0 admits a flat unitary connection. (This is essentially the original proof of Narasimhan and Seshadri.) For simplicity we consider the case of rank 2. Then if E is polystable but not stable it is a sum of degree 0 line bundles and the statement follows immediately from the line bundle discussion. So suppose $E = E_1$ is stable. We assume known that there is *some* irreducible flat connection, defining a stable bundle E_0, and that E_0, E_1 can be connected by a real 1-parameter family E_t for $t \in [0,1]$ with E_t stable for all t. (We will come back to this later.) So we want to show that the set $S \subset [0,1]$ of parameter values t for which E_t admits a flat unitary connection is **open** and **closed**.

Closedness Suppose $t_i \in S$ converging to $t_\infty \in [0,1]$ and write $E_i = E_{t_i}$. The set of equivalence classes of flat unitary connections is obviously compact (viewed as representations of π_1 in a compact group). Passing to a subsequence, we can suppose that the flat connections corresponding to E_i converge to some limiting flat connection. These flat connections define a sequence of holomorphic bundles F_i $(i \leq \infty)$ with $F_i \to F_\infty$ in an obvious sense as $i \to \infty$. By definition, F_i is isomorphic to E_i for $i < \infty$ so in particular we have nonzero holomorphic maps

$$\phi_i : E_i \to F_i \qquad i < \infty.$$

We can normalise these maps with respect to a suitable sequence of metrics to have (say) L^2 norm 1. It is then an easy result from analysis (which fits into the general slogan "semi-continuity of cohomology") that there is some non-trivial holomorphic map $\phi_\infty : E_\infty \to F_\infty$. We now invoke an important property of stable bundles:

If E, F are stable bundles of the same rank and degree then any non-trivial holomorphic map from E to F is an isomorphism.

The proof of this property (in the rank 2, degree 0 case) goes as follows. By general principles, any nontrivial holomorphic map $\psi : E \to F$ which is not an isomorphism factors as

$$E \xrightarrow{p} L_1 \xrightarrow{\tilde{\psi}} L_2 \xrightarrow{\iota} F,$$

where p is projection to a quotient bundle and ι is the inclusion of a sub-bundle. The stability of E implies that $\deg(L_1) > 0$ and the stability of F implies that $\deg(L_2) < 0$. This is a contradiction since $\tilde{\psi}$ is a nontrivial section of the line bundle $L_2 \otimes L_1^*$ which has strictly negative degree.

Using this, we complete the proof of closedness. We know that E_∞ is stable and F_∞ is polystable (since it has a flat unitary connection). If F_∞ is stable then the non-trivial map ϕ_∞ and the property above show that E_∞ is isomorphic to F_∞, hence E_∞ has a flat unitary connection and $t_\infty \in S$. If F_∞ were polystable but not stable it would be a sum of degree zero line bundles and ϕ_∞ has a non-trivial component mapping to one of these. But by the same principle as above this component factors through a quotient of E_∞ of negative degree, contradicting the stability of E_∞.

Openness

We give an (inadequate) informal treatment based on the linearisation of the zero curvature condition and the action of the gauge group, i.e the *deformation theories* of flat connections and holomorphic bundles.

Flat unitary connections

Let $\mathcal{A}$ be the space of connections on a C^∞ $U(r)$ bundle E (an infinite dimensional affine space). The moduli space of flat connections is $F^{-1}(0)/\mathcal{G}$ where $\mathcal{G}$ is the group of unitary automorphisms of E and F is the curvature map.

If A_0 is a flat unitary connection we have a complex:

$$\Omega^0(\mathrm{ad}E) \xrightarrow{d_{A_0}} \Omega^1(\mathrm{ad}E) \xrightarrow{d_{A_0}} \Omega^2(\mathrm{ad}E).$$

This is the de Rham complex twisted by the flat bundle (or local co-efficient system) $\mathrm{ad}E$ and the cohomology groups $H^*(A_0)$ can be seen as the cohomology of the sheaf of locally constant sections of this bundle. The cohomology $H^1(A_0)$ represents solutions of the linearisation of the equation $F(A_0 + a) = 0$ modulo infinitesimal gauge transformations. Assuming for simplicity here that A_0 is irreducible this $H^1_{A_0}$ is the tangent space of the moduli space of flat connections at $[A_0]$.

Holomorphic bundles

Now we view the space of connections as the space of $\overline{\partial}$-operators. The moduli set of holomorphic bundles is $\mathcal{A}/\mathcal{G}^c$ where $\mathcal{G}^c$ is the group of general linear automorphisms of E. The linearisation of the action is

$$-\overline{\partial}_{A_0} : \Omega^0(\mathrm{ad}E^c) \to \Omega^{0,1}(\mathrm{ad}E^c).$$

Here $\mathrm{ad}E^c$ can be identified with the bundle $\mathrm{End}E$ which is now viewed as a holomorphic bundle, using the structure defined by A_0. Ignoring the technicalities about the meaning of the "moduli space", we see, roughly, that its tangent space at the corresponding point is the cokernel of $\overline{\partial}_{A_0}$, which is the Dolbeault representation of the sheaf cohomology group $H^1(\mathrm{End}E)$. (A better statement is that the *versal deformation space* of E as a holomorphic bundle is a neighbourhood of 0 in this H^1.)

The key to the openness is that there is a Hodge theory isomorphism

$$H^1(A_0) = H^1(\mathrm{End}E).$$

This is proved just like in the case of the trivial line bundle at the beginning of this section. One uses this to show that any small deformation of the holomorphic structure can be matched with a small deformation of the flat unitary connection.

The theorem of Narasimhan and Seshadri can be seen as an instance of a general relation between "complex and symplectic quotients" which we will now review.

Let (X, Ω) be a symplectic manifold and H a function on X. There is a unique vector field v such that $i_v\Omega = dH$ and this generates a flow of symplectomorphisms. For simplicity here, restrict to the case that this defines an S^1-action on X. If $t \in \mathbf{R}$ is not a critical value of H then $H^{-1}(t)$ is a submanifold of X. The action of S^1 on this submanifold has at most finite stabilisers and the quotient $Y = H^{-1}(t)/S^1$ is an orbifold. For simplicity suppose that it is actually a manifold. The key point is that there is an induced symplectic form on Y–it is the "symplectic quotient" of X.

The generalisation to a Lie group K acting on (X, Ω) is a *moment map*

$$\mu : X \to \mathrm{Lie}(K)^*.$$

That is, μ combines the Hamiltonians for all 1-parameter subgroups of X. We also suppose that μ is an equivariant map, intertwining the K actions on X and $\mathrm{Lie}(K)^*$. The symplectic quotient is

$$Y = \mu^{-1}(0)/K$$

and it has an induced symplectic form. (If the Lie algebra of K has a non-trivial centre there is the possibility of varying the construction with a parameter like t above.)

Now suppose that X is a Kähler manifold and K acts by holomorphic isometries. Suppose that there is a complexification K^c of K; this will also act holomorphically on X, but not preserving the symplectic form. Then the general idea is that there should be an open K^c-invariant subset $X^s \subset X$ of "stable points" (we ignore here the distinction with polystable) such that the *complex quotient* X^s/K^c can be identified with the *symplectic* quotient $\mu^{-1}(0)/K$.

Example 1. Take the standard S^1 action on $\mathbf{C}^n$. For any fixed t a moment map is given by $|z|^2 - t$. For $t < 0$ the symplectic quotient is empty (so there are no stable points). For $t > 0$ the quotient is

$$\mathbf{CP}^{n-1} = S^{2n-1}/S^1.$$

Then we should take $X^s = \mathbf{C}^n \setminus \{0\}$ and the complex quotient gives the other description

$$\mathbf{CP}^{n-1} = (\mathbf{C}^n \setminus \{0\})/\mathbf{C}^*.$$

Example 2. Take the S^1 action on $\mathbf{C}^p \times \mathbf{C}^q$ with weight 1 on the first factor and weight -1 on the second. Then for any fixed t we have a moment map

$$\mu(z_1, z_2) = |z_1|^2 - |z_2|^2 - t.$$

If $t > 0$ the stable points are those with $z_2 \neq 0$ and if $t < 0$ those with $z_1 \neq 0$.

The relevance to our preceding discussion is that $\mathcal{A}$ has a natural symplectic form and the curvature gives a moment map for the action of the gauge group $\mathcal{G}$, so the moduli space of flat connections is the symplectic quotient. The theorem of Narasimhan and Seshsadri asserts that indeed this agrees with the complex quotient if $\mathcal{A}^s \subset \mathcal{A}$ is the subset defining polystable holomorphic bundles.

What we conclude from this is that there is a moduli space which is a complex object (regarded as a moduli space of polystable holomorphic bundles) but also has an induced symplectic form. This symplectic form is given by the wedge product on $H^1(\mathrm{ad}E)$ induced by

$$\Omega(a, b) = \frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}(a \wedge b).$$

(The factor being included to fit in with later conventions.)

The discussion above is imprecise because the moduli space as we have defined it will not be a smooth manifold–there are singularities at the polystable points which correspond to reducible connections (or reducible representations $\pi_1 \to U(r)$). But there is a variant which avoids this difficulty.

Stability and polystability are defined for bundles of any degree, so the question is how to modify the notion of a flat connection. There are group homomorphisms

$$U(r) \to PU(r) \quad U(r) \to S^1,$$

where the first is the quotient map to $PU(r) = U(r)/$diagonal matrices and the second is the determinant. At the Lie algebra level, $U(r)$ splits as a product but at the group level this is only up to a finite cover. A connection on a $U(r)$ bundle E induces connections on the corresponding $PU(r)$ and S^1 bundles. Fix a real 2-form ρ on Σ with integral -2π. Then we can consider connections on E which

- Induce a flat $PU(r)$ connection;
- Induce a connection on the determinant bundle $\Lambda^r E$ with curvature $id\ \rho$ where d is the degree of E.

The general version of the Narasimhan-Seshadri result gives a correspondence between such connections and polystable bundles. The proofs are not essentially different from the previous case ($d = 0$) because all the interesting geometry occurs in the $PU(r)$ factor.

If d, r are coprime then it is easy to show that any polystable bundle is stable and there are no reducible connections of this kind. Then we get a compact, smooth moduli space. Another variant is to fix the induced connection on $\Lambda^r E$: this corresponds to holomorphic bundles with a fixed holomorphic structure on $\Lambda^r E$. From now on we restrict to the simplest case, when $r = 2, d = 1$. We get a fixed-determinant moduli space $\mathcal{N}(\Sigma)$ which is in fact a complex projective manifold of complex dimension $3g - 3$, where g is the genus of Σ. We can write this down very explicitly by taking ρ to be supported in a small disc D about a point $p \in \Sigma$. The connections can be taken to be flat $SU(2)$ connections over $\Sigma \setminus D$ with holonomy $-1 \in SU(2)$ around the boundary of D. Taking the standard description of $\pi_1(\Sigma)$, the moduli space $\mathcal{N}(\Sigma)$ can be realised as systems of $2g$-matrices $A_1, \ldots, A_g, B_1, \ldots, B_g \in SU(2)$ satisfying the equation

$$\prod_{i=1}^{g} [A_i, B_i] = -1 \tag{1}$$

modulo conjugation by $SU(2)$. (Here $[A, B]$ is the group commutator.)

Note that the dimension checks with a naive count: $SU(2)$ is 3-dimensional so there are $3.2g$ parameters in specifying the A_i, B_i and we lose 3 when imposing the equation and 3 when dividing by conjugation; so we end up with $6g - 6$ real dimensions.

Examples for low genus

For genus 0 the moduli space $\mathcal{N}$ is empty. For genus 1 it is a single point. Algebro-geometrically we have a fixed line bundle $\Lambda \to \Sigma$ of degree 1 and we claim that there is just one stable rank 2 bundle E over Σ with $\Lambda^2 E = \Lambda$. Standard results for curves of genus 1 show that $H^1(\Lambda^*) = \mathbf{C}$. So there is a unique non-trivial extension

$$0 \to \mathcal{O} \to E \to \Lambda \to 0. \tag{2}$$

To see that E is stable we need to see that for any line bundle L of degree > 0 we have $H^0(E \otimes L^*) = 0$. This follows from the exact sequence in cohomology using the fact that the map

$$H^0(\Lambda \otimes L^*) \to H^1(L^*)$$

is the cup product with the extension class. So we have found one stable bundle. If F is any stable bundle of the required type then Riemann-Roch shows that $H^0(F)$ is non-trivial. If s is a non-trivial section then s cannot vanish anywhere because if it did the section would factor through a line subbundle of degree > 0 which would contradict stability. So s gives a subbundle inclusion $\mathcal{O} \to F$ and the determinant condition means that the quotient bundle is Λ. Thus F is an extension like (2) and we deduce that F is isomorphic to E.

From the point of view of flat connections we can write down a pair $A, B \in SU(2)$ satisfying $[A, B] = -1$:

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

And you can check that up to conjugation this is the only solution of (1) when $g = 1$.

Now suppose that Σ has genus 2. The moduli space $\mathcal{N} = \mathcal{N}(\Sigma)$ is then an intersection of two quadric hypersurfaces in $\mathbf{CP}^5$. We will not prove this but we will write down a family of bundles parametrised by this space. In fact we will just write down the family of projectivized bundles, with fibre $\mathbf{P}^1$.

Recall some theory of quadrics. If Q is a smooth quadric hypersurface in $\mathbf{P}(V)$, defined by a quadratic form q on V, then the tangent space to Q at a point $[v] \in Q$ can be identified with $v^\perp/v$ where $v^\perp$ is the orthogonal subspace to v with respect to q. There is a natural induced quadratic form on $v^\perp/v$ which leads to a "holomorphic conformal structure" on Q (a nonsingular quadratic form on each tangent space, defined up to constant factor). So at each point $p \in Q$ we have a quadric hypersurface in $\mathbf{P}(TQ)_p$. Recall that a nonsingular quadric in $\mathbf{P}^3$ is isomorphic to a product $\mathbf{P}^1 \times \mathbf{P}^1$. So if Q is 4-dimensional we get a pair of $\mathbf{P}^1$-bundles, say $\mathbf{P}^+, \mathbf{P}^- \to Q$. But there is no intrinsic way to distinguish between $\mathbf{P}^\pm$: we get a 2-fold covering of the space of nonsingular quadrics given by the choice of one of these bundles.

Our genus 2 Riemann surface Σ is a double cover : $p : \Sigma \to \mathbf{P}^1$ branched at 6 points $\lambda_1, \ldots, \lambda_6$. We choose a corresponding pencil of quadrics Q_λ in $\mathbf{P}^5$. The quadric Q_λ is defined by a form $q_\lambda = q_0 + \lambda q_\infty$ on $\mathbf{C}^6$ where $q_0 = -\sum \lambda_i z_i^2$ and $q_\infty = \sum z_i^2$. We restrict the covering above to this pencil of quadrics to get a 2-fold covering of $\mathbf{P}^1 \setminus \{\lambda_i\}$, which can be identified with Σ punctured at the branch points. One checks that this extends over the branch points, so for each $\sigma \in \Sigma$ we have a $\mathbf{P}^1$-bundle $\mathbf{P}^{(\sigma)}$ over the smooth locus of $Q_{p(\sigma)}$. Now let $\mathcal{N}$ be the common intersection of all the Q_λ. We have a $\mathbf{P}^1$ bundle over $\mathcal{N} \times \Sigma$ with fibre $\mathbf{P}_v^{(\sigma)}$ over $([v], \sigma)$. For any point in $\mathcal{N}$ we have a $\mathbf{P}^1$ bundle over Σ. These can be lifted to vector bundles of degree 1 with fixed determinant and the statement is that these are exactly the stable bundles.

Cohomology pairings on $\mathcal{N}(\Sigma)$.

There is an important general construction for producing cohomology classes over moduli spaces of bundles (or connections). In the case at hand, with the moduli space $\mathcal{N} = \mathcal{N}(\Sigma)$ we have a universal $U(2)$ bundle $\mathbf{E} \to \mathcal{N} \times \Sigma$. (In general there are subtleties involved in the construction of such a universal bundle, but we ignore these here: the subtleties do not affect rational cohomology which is what we use.) Now we take the characteristic class

$$\chi(\mathbf{E}) = (c_2 - \frac{1}{4}c_1^2)(\mathbf{E}) \in H^4(\mathcal{N} \times \Sigma)$$

and decompose this according to Künneth components. This can be expressed by saying that we have maps

$$H_i(\Sigma) \to H^{4-i}(\mathcal{N}).$$

So we get classes $a \in H^2(\mathcal{N}), b \in H^4(\mathcal{N})$ and a map

$$m : H_1(\Sigma) \to H^3(\mathcal{N}).$$

The class a is the same as the de Rham cohomology class of the symplectic form on $\mathcal{N}$.

If we take any product Π of these classes to get up to the top dimension $6g-6$ we can evaluate on the fundamental class to get a number $\langle \Pi, \mathcal{N} \rangle$. We will focus here on the most important case when $\Pi = a^{3g-3}/(3g-3)!$. The pairing then has an interpretation as the *volume* of $\mathcal{N}$ using the volume form defined by the symplectic structure. From now on we will write $\mathcal{N}_g$, where g is the genus of Σ.

Explicit formulae for these volumes were found (about the same time) by Thaddeus and Witten. Witten's formula is

$$\mathrm{Vol}(\mathcal{N}_g) = \frac{2}{(2\pi^2)^{g-1}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2g-2}}. \qquad (3)$$

As we will review below, the infinite sum can be evaluated explicitly in terms of "Bernoulli numbers".

Many approaches to these questions have been developed. We will outline an approach based on the symplectic geometry of S^1-actions.

Some general theory

Let (X, ω) be a symplectic manifold and H a proper function on X which generates an S^1-action. The properness condition implies that the symplectic quotients $M_t = H^{-1}(t)/S^1$ are compact and the critical values of H form a discrete subset of $\mathbf{R}$. Recall that these critical values correspond to fixed points of the S^1-action on X. Let $V(t)$ be the volume of M_t, computed using the induced symplectic form. Then we have the following general properties.

- The push forward by H of the volume form on X is $V(t)dt$.
- $V(t)$ is a piecewise polynomial function, smooth away from the critical values. At a critical value corresponding to a component K of the fixed set the jumps in the derivatives of V can be computed from data localised on K. (More precisely, from the normal bundle of K in X, regarded as a bundle with an S^1 action.) If K has codimension $2k$ the derivative $V^{(i)}$ are continuous for $i < k-1$ and have a jump discontinuity for $i = k-1$. The possible discontinuities for $i \geq k$ depend on the more detailed topology of the normal bundle.

We do not take time here to go into the detailed theory for computing these discontinuities.

The piecewise polynomial behaviour of $V(t)$ away from the critical values can be seen as follows. If the parameter t does not cross a critical value the symplectic quotient M_t does not change up to diffeomorphism, so we can regard the symplectic forms ω_t as defined on the same manifold. We regard $H^{-1}(t) \to M_t$ as a principal circle bundle P over M_t and one finds that

$$\frac{d}{dt}[\omega_t] = c_1(P) \in H^2(M_t).$$

So $[\omega_{t_0+\tau}] = [\omega_{t_0}] + \tau c_1(P)$ and the top power of $[\omega_{t_0+\tau}]$ is a polynomial function of τ.

Examples

1. Take the standard S^1 action on $\mathbf{C}^n$ with Hamiltonian $H(z) = |z|^2$. Then $V(t) = 0$ for $t < 0$ and is proportional to t^{n-1} for $t > 0$, so we see the jump discontinuity in the $(n-1)st.$ derivative.

2. Take S^1 acting by rotations on the 2-sphere. The Hamiltonian is the projection to the axis of rotation. The symplectic quotients are either empty or single points and $V(t) = 1$ for $-1 \leq t \leq 1$ and 0 for $|t| > 1$. (This is with a symplectic form on S^2 normalised to have area 2.)

Now return to our moduli spaces. Let $\overline{\Sigma}$ be a surface of genus $g+1$ and $\gamma \subset \overline{\Sigma}$ an embedded non-separating circle. Let Σ be the surface of genus g obtained by cutting $\overline{\Sigma}$ along γ and gluing in a pair of discs. So conversely we have a pair of points p_1, p_2 in Σ and we construct $\overline{\Sigma}$ by cutting out discs centred at these points and gluing the resulting boundaries. Let $\mathcal{N}_{g+1}$ be the moduli space associated to $\overline{\Sigma}$ and $\mathcal{N}_g$ to Σ.

Recall that $\mathcal{N}_{g+1}$ is given by conjugacy classes of solutions of the equation

$$\prod_{i=1}^{g+1} [A_i, B_i] = -1.$$

We can suppose that A_{g+1} corresponds to the holonomy of a connection around γ. We define a function $h : \mathcal{N}_{g+1} \to [0,1]$ by

$$h = \frac{1}{\pi} \cos^{-1} \left(\mathrm{Tr}(A_{g+1})/2 \right).$$

So $h = t$ if A_{g+1} is conjugate to

$$\begin{pmatrix} e^{\pi it} & 0 \\ 0 & e^{-\pi it} \end{pmatrix}.$$

Note that there is a symmetry of $\mathcal{N}_{g+1}$ which multiplies A_{g+1} by -1 and this covers the symmetry $t \to 1-t$ of $[0,1]$. For $i = 0, 1$ let $W_i = h^{-1}(i)$. Then h is smooth and proper on $\mathcal{N}_{g+1} \setminus (W_0 \cup W_1)$. The essential fact is that h is the Hamiltionian of a circle action on $\mathcal{N}_{g+1} \setminus (W_0 \cup W_1)$. The circle action can be written down as follows. A point $A \in SU(2)$ not equal to ± 1 lies on a unique 1-parameter subgroup: we define $L(A)$ in the Lie algebra to be the standard generator of this subgroup, normalised so that $\exp(L(A)) = 1$. Now we get a circle action on $\mathcal{N}_{g+1} \setminus (W_0 \cup W_1)$ induced by the maps which takes B_{g+1} to $B_{g+1}\phi_s$ where $\phi_s = \exp(sL(A_{g+1}))$. Since ϕ_s commutes with A_{g+1} this preserves the commutator $A_{g+1}B_{g+1}A_{g+1}^{-1}B_{g+1}^{-1}$. The statement (whose proof is not hard but which we omit) is that h generates this circle action.

The fixed points of this circle action arise as follows. If

$$A_{g+1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad B_{g+1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then

$$\phi_s = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

with $\lambda = e^{2\pi s}$. Then

$$B_{g+1}\phi_s = \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix} = \phi_s^{-1/2} B_{g+1} \phi_s^{1/2},$$

with the obvious meaning of $\phi_s^{1/2}$. Then if A_i, B_i for $1 \leq i \leq g$ are any collection of diagonal matrices in $SU(2)$ we get a point in $\mathcal{N}_{g+1}$ which is fixed by the circle action. One finds that these are the only fixed points. The conclusion is that the S^1 action on $\mathcal{N}_{g+1} \setminus (W_0 \cup W_1)$ has a fixed set $J \subset h^{-1}(1/2)$ which is a $(2g)$-dimensional torus (a copy of the Jacobian of Σ), and is otherwise free.

As before, let V be the function on $[0,1]$ corresponding to this circle action. We claim that it has the following properties.

1. $V(t) = V(1-t)$.
2. $V(0) = 0$ and for $0 \leq t < 1/2$ the function $V(t)$ is an *even* polynomial in t.
3. There is a jump discontinuity, $-\kappa$ say, in the $(2g-1)$'st derivative of V at $t = 1/2$ but all other derivatives are continuous.

It is an elementary exercise to see that *these properties completely determine* $V(t)$. If we extend V from $[0,1]$ to a $\mathbf{Z}$-periodic function $\tilde{V}$ on $\mathbf{R}$, or equivalently a function on $\mathbf{R}/\mathbf{Z}$, it is characterised as the unique solution of the equation

$$\left(\frac{d}{dt}\right)^{2g} \tilde{V} = \kappa(1 - \delta_{1/2}), \tag{4}$$

with $\tilde{V}(0) = 0$.

The first property above follows from the symmetry of $\mathcal{N}_{g+1}$ covering $t \mapsto 1-t$. The third property follows from considerations of the normal bundle of J in $\mathcal{N}_{g+1}$. J has codimension $6g - 2g = 4g$ and one finds that the normal bundle is trivial (viewed as a complex vector bundle with complex structure defined by the circle action) which means that there are no jumps in the higher derivatives.

To see the second property we develop a more explicit description of the symplectic quotients for small t. The fibre $W_0 = h^{-1}(0)$ is a fibre bundle with fibre $S^3 = SU(2)$ over $\mathcal{N}_g$. Geometrically this corresponds to a choice of "gluing parameter" identifying the fibres at p_1, p_2 of a flat bundle over Σ. This fibre W_0 has codimension 3 in $\mathcal{N}_{g+1}$ and the boundary of a tubular neighbourhood of W_0 is an $S^3 \times S^2$ bundle over $\mathcal{N}_g$. Dividing by the circle action, we get an $S^2 \times S^2$ bundle over $\mathcal{N}_g$. The statement which emerges from this analysis is that for small $t > 0$ the symplectic quotient is the total space of an $S^2 \times S^2$ bundle $\pi : Q \to \mathcal{N}_g$ and the class of the symplectic form is

$$[\omega_t] = \pi^* a + t(h_1 + h_2) \tag{5}$$

where $a \in H^2(\mathcal{N}_g$ is the class we defined before and h_1, h_2 are 1/2 times the first Chern classes of the tangent bundle along the two S^2 factors in the fibre. These classes satisfy

$$h_i^2 = \pi^*(b) \in H^4(Q), \tag{6}$$

where again $b \in H^4(\mathcal{N}_g)$ is the class we defined before.

If one grants these statements, the function $V(t)$ for small t (and in fact for $t < 1/2$) can be found by straightforward calculations in the cohomology ring of Q. Given a class $\Theta \in H^{6g-g}(\mathcal{N}_g)$ we have an equality of pairings

$$\langle \Theta h_1 h_2, Q \rangle = \langle \Theta, \mathcal{N}_g \rangle \tag{7}$$

So to evaluate

$$V(t) = \frac{1}{6g!} \langle (\pi^*(a) + t(h_1 + h_2))^{6g}, Q \rangle$$

we expand the power, use the relations $h_i^2 = \pi^* b$ to elminate higher powers of the h_i and the relation (7) to express eveything in terms of the pairings

$$\langle a^{3g-3-2p} b^p, \mathcal{N}_g \rangle. \tag{8}$$

It is clear from this that $V(0) = 0$ and that only even powers of t contribute. Moreover one finds that all the pairings (8) can be determined from the coefficients in the polynomial function V. We just consider the simplest case which is the t^2 coefficient and one gets simply:

$$V(t) = \mathrm{Vol}(\mathcal{N}_g) t^2 + O(t^4). \tag{9}$$

On the other hand we know that

$$\mathrm{Vol}(\mathcal{N}_{g+1}) = \int_0^1 V(t) dt. \tag{10}$$

We will now write down the solution of the equation (4) in two ways, which will give formulae for our volumes.

Define a sequence of polynomials $R_m(t)$ for $m = 0, 1, 2, \ldots$ inductively by

- $R_0(t) = 1$;
- $R'_m = R_{m-1}$ for $m \geq 1$;
- $\int_{-1/2}^{1/2} R_m(t)dt = 0$ for $m \geq 1$.

Then a moments thought shows that if we extend $R_m|_{(-1/2,1/2)}$ to a **Z**-periodic function $\tilde{R}$ on **R** we get a solution of

$$\left(\frac{d}{dt}\right)^m \tilde{R} = 1 - \delta_{1/2}.$$

So we have

$$\tilde{V}(t) = \kappa(\tilde{R}_{2g}(t) - \tilde{R}_{2g}(0)).$$

It follows from (9), (10) that

$$\mathrm{Vol}(\mathcal{N}_{g+1}) = -\kappa R_{2g}(0).$$

while $\mathrm{Vol}(\mathcal{N}_g)$ is κ times the t^2 co-efficient of R_{2g}.

Define numbers c_i by

$$\frac{u}{\sinh u} = c_0 + c_1 u + c_2 u^2 + \ldots.$$

So $c_0 = 1$ and the c_i are determined inductively by

$$c_p + \frac{1}{3!}c_{p-1} + \frac{1}{5!}c_{p-2} \ldots + \frac{1}{(2p+1)!} = 0.$$

In fact

$$c_p = (-1)^p(2^{2p} - 2)\frac{B_p}{(2p)!},$$

where B_p are the Bernoulli numbers.

Set $r_m = 2^{2m}R_{2m}(0)$. Then the inductive definition implies that

$$R_{2m}(t) = 2^{-2m}r_m + 2^{-2(m-1)}r_{m-1}\frac{t^2}{2!} + \ldots$$

and the vanishing of the integral of R_{2m} from $-1/2$ to $1/2$ requires

$$r_m + \frac{r_{m-1}}{3!} + \ldots = 0.$$

So we see that $r_m = c_m$. Thus

$$\frac{\mathrm{Vol}(\mathcal{N}_{g+1})}{\mathrm{Vol}(\mathcal{N}_g)} = \frac{-1}{2}\frac{c_g}{c_{g-1}}.$$

Hence

$$\mathrm{Vol}(\mathcal{N}_g) = Z\left(\frac{-1}{2}\right)^g c_{g-1}$$

for some number Z, independent of g. Taking $g = 1$ we see that $Z = -2$ so we have an explicit volume formula

$$\mathrm{Vol}(\mathcal{N}_g) == \left(\frac{-1}{2}\right)^{g-1} c_{g-1}.$$

(This means that $\kappa = -1^{g-1}2^g$, as can be shown directly by considerations of the geometry of $J \subset \mathcal{N}_{g+1}$.)

For the second approach we use Fourier theory to see that

$$\tilde{R}_{2g}(t) = -\sum_{k\in\mathbf{Z},k\neq 0} \frac{e^{2\pi ik(t+1/2)}}{(2\pi ik)^{2g}}.$$

Evaluating at 0 this leads to Witten's formula (3)

While the moduli spaces of flat $SU(2)$ bundles are singular they have well-defined volumes. In that case Witten has a formula which is the same as (3) but omitting the alternating sign $(-1)^{n+1}$. For a general compact Lie group G with finite centre $Z(G)$, Witten gives a formula for the volume of the moduli space of flat G-connections:

$$\mathrm{Vol}(\mathcal{M}) = \frac{\mathrm{Vol}(G)^{2g-2}|Z(G)|}{(2\pi)^{\dim\mathcal{M}}}\sum_{\alpha}\frac{1}{(\dim\,\alpha)^{2g-2}}$$

where α runs over the irreducible representations of G.

One of the reasons for being interested in these volumes is the connection with *Verlinde Theory*. Going back to $\mathcal{N}(\Sigma)$, there is a positive holomorphic line bundle $L \to \mathcal{N}(\Sigma)$ with $c_1(L) = a$. Thus we have vector spaces

$$V_k(\Sigma) = H^0(L^k).$$

The leading term in the Riemann-Roch formula says that as $k \to \infty$

$$\dim V_k = k^{3g-3}\mathrm{Vol}(\mathcal{N}) + O(k^{3g-2}).$$

In fact an examination of the Riemann-Roch formula shows that knowledge of all the dimensions of the V_k is equivalent to knowledge of all the pairings (8).

3 Dimension 3

The distinctive structure present in gauge theory in 3 dimensions is the *Chern-Simons functional*. Let M be compact oriented 3-menifold and consider $SU(r)$ connections on a bundle $P \to M$.

First approach
We define a 1-form Θ on the space of connections $\mathcal{A}$ by

$$\Theta_A(a) = \int_M \mathrm{Tr}(F_A \wedge a).$$

We claim that this a closed 1-form. This amounts to the fact that the derivative

$$(\delta\alpha)(a,b) = \int_M \mathrm{Tr}(d_A b \wedge a).$$

Is symmetric in a, b. Also, α vanishes on the tangent vectors to the orbits of the gauge group $\mathcal{G} = \mathrm{Aut}P$. This is the identity

$$\int_M \mathrm{Tr}(F_A \wedge d_A u) = 0,$$

which follows from the Bianchi identity $d_A F_A = 0$.

It follows that, at least locally in the space of connections, Θ is the derivative of a $\mathcal{G}$-invariant function. We normalise by a factor $4\pi^2$ and define the Chern-Simons functional to be the function with derivative $(4\pi^2)^{-1}\Theta$. In this case the constant ambiguity can be fixed by saying that the function is 0 on the trivial connection.

Second approach
Let A_0, A_1 be two connections on P. We pull-back P to the cylinder $M \times [0,1]$ and consider a connection $\mathbf{A}$ over the cylinder with boundary values A_0, A_1. Of course $\mathbf{A}$ is not unique but the Chern-Weil integral

$$I(\mathbf{A}) = \frac{1}{8\pi^2} \int_{M\times[0,1]} \mathrm{Tr}(F_{\mathbf{A}}^2)$$

depends on $\mathbf{A}$ only up to an integer. This follows from a gluing argument and Chern-Weil theory on the 4-manifold $M \times S^1$. So we define

$$CS(A_1) - CS(A_0) = I(\mathbf{A}) \in \mathbf{R}/\mathbf{Z},$$

and we have a Chern-Simons functional taking values in $\mathbf{R}/\mathbf{Z}$.

Third approach
In fact the bundle $P \to M$ is trivial so we can choose a trivialisation and regard the connection as a matrix valued 1-form A. Then we have an explicit formula

$$CS(A) = \frac{1}{4\pi^2} \int_M \mathrm{Tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A\right),$$

and this is independent of the trivialisation up to an integer.

The critical points of CS are the flat connections on P. We want to consider an invariant of M given by "counting" these flat connections, formally analogous to the Euler characteristic of $\mathcal{A}/\mathcal{G}$. This is the *Casson invariant*, which is defined in the case when M is a homology 3-sphere.

Approximate definition *The Casson invariant $C(M)$ of a homology 3-sphere M is half the number of irreducible critical points of a small generic perturbation of the Chern-Simons functional, counted with suitable signs.*

We give a brief sketch of some of the technology required to make sense of this (and similar constructions).

- The set of equivalence classes of flat connections is clearly *compact.*
- The deformation theory of flat connections about some A_0 can be set up using the twisted de Rham complex:
$$\Omega^0(\mathrm{ad}P) \stackrel{d_{A_0}}{\to} \Omega^1(\mathrm{ad}P) \stackrel{d_{A_0}}{\to} \Omega^2(\mathrm{ad}P) \stackrel{d_{A_0}}{\to} \Omega^3(\mathrm{ad}P). \quad (11)$$
There is a Poincaré duality $H^{3-i}_{A_0} = \left(H^i_{A_0}\right)^*$ so everything is determined by $H^0_{A_0}$ and $H^1_{A_0}$.
- $H^0_{A_0}$ is the Lie algebra of the group Γ_{A_0} which is the stabiliser of A_0 in $\mathcal{G}$. The connection is irreducible if $\Gamma_{A_0} = \{\pm 1\}$ and $H^0_{A_0} = 0$.
- Let $d^*_{A_0} : \Omega^1(\mathrm{ad}P) \to \Omega^0(\mathrm{ad}P)$ be the formal adjoint of d_{A_0}. For suitable small ϵ the set
$$T_{A_0} = \{A_0 + a : d^*_{A_0} a = 0, \|a\| < \epsilon\}$$
is a Γ_{A_0}-equivariant slice for the action of $\mathcal{G}$. A neighbourhood of $[A_0] \in \mathcal{A}/\mathcal{G}$ is modelled on T_{A_0}/Γ_{A_0}.
- The Chern-Simons functional fits into a "nonlinear Fredholm package" which means that many constructions from finite-dimensional differential topology extend. For example, it can be shown that there is a Γ_{A_0}-invariant function on the finite dimensional space $H^1_{A_0}$
$$f : H^1_{A_0} \to \mathbf{R}$$
and a Γ_{A_0}-equivariant diffeomorphism from T_{A_0} to a neighbourhood of 0 in a product $H^1_{A_0} \times V$, where V is a Hilbert space, taking the Chern-Simons functional to $f + Q$ where Q is a non-degenerate quadratic form on V. Locally, near A_0, we could perturb CS by perturbing f to a nearby *Morse function.*
- In the case at hand, when M is a homology sphere, the only irreducible flat connection is the trivial one.

- In general *reducible solutions cause fundamental difficulties*. This can seen by the example of the S^1-invariant functions on $\mathbf{C}$:

$$f_\epsilon(z) = |z|^4 + \epsilon|z|^2.$$

  Taking small positive or negative ϵ does not essentially change the function for $|z|$ large. But there is a critical orbit for $\epsilon < 0$ which disappears for $\epsilon > 0$.

- There is theory required to discuss signs and orientations, but we will not go into that here.

Casson's original approach was different. Any 3-manifold M has a Heegard decomposition $M = W_+ \cup_\Sigma W_-$ where $W_\pm$ are handlebodies and Σ is a surface. Then we have subsets $L_\pm \subset \mathcal{M}(\Sigma)$ in the moduli space of flat $SU(2)$ connections over Σ representing connections which extend to flat connections over $W_\pm$. The Casson invariant is half the intersection number of L_+, L_- (in the smooth part of $\mathcal{M}$ representing irreducible connections).

Surgery

Let $K \subset M$ be a knot in a homology 3-sphere and N be a tubular neighbourhood of the knot: a solid torus. The boundary ∂N is a torus with a standard basis l, m for π_1:

- the meridian m bounds a disc in N;
- the longitude l bounds in homology in $M \setminus N$.

We construct the 0-surgered manifold M_0 by re-attaching the solid torus so that l (regarded as living in $M \setminus N$) bounds a disc in the solid torus and the 1-surgered manifold M_1 so that $l - m$ bounds.

M_1 is again a homology sphere while M_0 is a homology $S^1 \times S^2$.

Casson's surgery formula is

$$C(M_1) = C(M) + \sum_j j^2 a_j \tag{12}$$

where a_j are the co-efficients of the normalised Alexander polynomial, whose theory we now recall in outline.

Let Λ be the ring of Laurent series with $\mathbf{Z}$-coefficients. It is the group ring of $\mathbf{Z}$. The units in Λ are $\pm t^\mu$. We have $H^1(M_0) = \mathbf{Z}$, so there is a $\mathbf{Z}$-cover $\tilde{M}_0 \to M_0$ and $H_1(\tilde{M}_0)$ is a module over Λ. It is a fact that it can be presented with an equal number of generators and relations, i.e. via a square matrix with entries in Λ. The determinant of this matrix is independent of the choice of presentation up to a unit: this is the Alexander polynomial of the knot. In fact it always has a symmetric form so we can normalise it to have the form

$$\Delta(t) = a_0 + \sum_{j\geq 1} a_j (t^j + t^{-j}),$$

and there is also a way to fix the ± 1 ambiguity. So the co-efficients a_j are invariants of the knot and these are what appear in (12).

The theory is easier to understand in the case of a "fibred" knot, when M_0 is the mapping torus of a diffeomorphism $f : \Sigma \rightarrow \Sigma$ for a surface Σ of genus g. Then $\Delta(t)$ is t^{-g} times the characteristic polynomial of the induced action on $H_1(\Sigma)$. The symmetry property follows from the fact that this preserves the intersection form on $H_1(\Sigma)$.

Casson's formula can be understood in two steps. First one defines an analogous invariant $C(M_0)$ of M_0, counting $U(2)$ connections which induce flat $PU(2)$ connections and with fixed determinant, just as we considered for surfaces. These can be identified with flat $SU(2)$ connections over $M \setminus K$ having holonomy -1 around l. There are no reducible solutions in this case. Then one shows

$$C(M_1) = C(M) + C(M_0); \tag{13}$$

and

$$C(M_0) = \sum j^2 a_j. \tag{14}$$

The formula (13) follows from intersection theory on the representation variety $\mathcal{M}$ of the torus ∂N: the moduli space of flat $SU(2)$ connections over ∂N. This is the quotient of the dual torus by ± 1, hence a 2-sphere. The equivalence classes of flat connections over $M \setminus N$ has "virtual dimension" 1 so after generic perturbation we can suppose it is a 1-dimensional set Γ. In fact it has boundary points at the reducible flat connections but it turns out that we can ignore these. There is a triangle in $\mathcal{M}$ such that the connections we are counting over M, M_1, M_0 correspond to the intersection of the image of Γ with the three sides of the triangle and this gives (13).

The formula (14) can be related to the homology $H_*(\mathcal{N}_g)$ of the moduli spaces we studied in Section 2, so we now digress to discuss that. There is an *Atiyah-Bott stratification* of $\mathcal{A}/\mathcal{G}$ consisting of an open set representing stable bundles, which retracts on to $\mathcal{N}_g$, and higher strata which retract on to copies of the Jacobian torus. In fact the higher strata have codimension $\geq 2g$. (Which gives one way of seeing that stable bundles can be joined by a path of stable bundles.)The rational cohomology of the irreducible connections modulo gauge is freely generated as a ring by the classes $a \in H^2, b \in H^4$ and the image of $m : H_1(\Sigma) \rightarrow H^3$. Atiyah and Bott showed that this stratificatiion gave a way to compute the Poincaré series of $\mathcal{N}_g$ (which can also be done by other methods). The formula is

$$\sum \dim H_i(\mathcal{N}_g) u^i = \frac{1}{1-u^2} \frac{1}{1-u^4} (1+u^3)^{2g} - \sum_{k \geq 0} \frac{(1+u)^{2g}}{(1-u^2)} u^{2g+4k}, \tag{15}$$

or

$$\frac{1}{(1-u^2)(1-u^4)} (1+u^3)^{2g} - u^{2g}(1+u)^{2g}. \tag{16}$$

The diffeomorphisms of Σ act on $\mathcal{N}_g$ and so on its homology. Using the Atiyah-Bott approach one can identify these representations in terms of the standard exterior powers $\Lambda^r = \Lambda^r H_1(\Sigma)$. The formula is

$$\sum H_i(\mathcal{N}_g) u^i = \frac{1}{(1-u^2)(1-u^4)} \sum_r \Lambda^r (u^{3r} - u^{2g+r}).$$

This can be tidied up by putting $\mathcal{H}_{(i)} = H_{3g-3-i}(\mathcal{N}_g)$ and $\Lambda_{(s)} = \Lambda^{g-s}$. Then we get

$$\sum \mathcal{H}_{(i)} u^i = \sum \frac{(u^{2j} - u^{-2j})(u^j - u^{-j})}{(u^2 - u^{-2})(u - u^{-1})} \Lambda_{(j)},$$

where the genus g has disappeared.

In particular forgetting the grading in the homology of $\mathcal{N}_g$ we have

$$\mathcal{H}_*(\mathcal{N}_g) = \sum j^2 \Lambda_{(j)}. \tag{17}$$

(More precisely, we should remember the $\mathbf{Z}/2$ grading: the $\Lambda_{(j)}$ for j even/odd appear in the either the even/odd regraded part of $\mathcal{N}_g$.)

If we have a diffeomorphism $f : \Sigma \to \Sigma$ the induced map on $\mathcal{N}_g$ has Lefschetz number $\pm(\sum (-1)^j \mathrm{Tr}(f_j)$ where $f_j : \mathcal{H}_{(j)} \to \mathcal{H}_{(j)}$ are the induced maps. On the other hand it follows from the definition that $a_j = (-1)^j \mathrm{Tr}\psi_j$ where $\psi_j : \Lambda_{(j)} \to \Lambda_{(j)}$ is the induced map. This can be used to prove (14) in the case of fibred knots.

A proof of (14) in the general case can be put in the framework of "Topological Field Theory". Recall that, in outline, a $(d+1)$-dimensional Topological Field Theory assigns to a closed oriented d-manifold S a vector space $V(S)$ and to an oriented $(d+1)$-dimensional manifold W with boundary an element $\Psi(W) \in V(\partial W)$. These assignments are required to satisfy some axioms, notably:

- The vector space associated to the empty set, regarded as a d-manifold, is the ground field: thus for a closed $d+1$ manifold W the invariant $\Psi(W)$ is a number.
- If $\overline{S}$ is the same manifold with reversed orientation then $V(\overline{S}) = V(S)^*$.
- For a disjoint union $S_1 \sqcup S_2$ we have
$$V(S_1 \sqcup S_2) = V(S_1) \otimes V(S_2).$$
- IF W_1, W_2 have the same boundary S with opposite orientations and we form a closed manifold W by gluing along S then
$$\Psi(W) = \langle \Psi(W_1), \Psi(W_2) \rangle.$$
Similarly for manifolds with several boundary components.

Often, as in our example below, this structure and axiom system is only realised approximately: for example there might be some restrictions on the manifolds considered. and there might be gradings introducing signs. If S_0, S_1

are d-manifolds and W is a cobordism from S_0 to S_1 we can regard $\Psi(W)$ as a linear map

$$\Psi(W) : V(S_0) \to V(S_1),$$

and this satisfies the obvious composition law. So we can think of the structure as a functor from d-manifolds and cobordisms to vector spaces and linear maps. If now S_0 is equal to S_1 so we can construct a closed d-manifold $\overline{W}$ by gluing the ends of the cobordism, then we have

$$\Psi(\overline{W}) = \mathrm{Tr}(\Psi(W)).$$

There is a very elementary construction of a structure along these lines. Let W be cobordiam from a surface Σ_0 to a surface Σ_1 (all with orientations). Then we have $H^1(W) \to H^1(\Sigma_0) \oplus H^1(\Sigma_1)$. The image defines a "Plucker point" in

$$\Lambda^*(H^1(\Sigma_0) \oplus H^1(\Sigma_1)) = \mathrm{Hom}(\Lambda^* H^1(\Sigma_0), \Lambda^* H^1(\Sigma_1)).$$

One finds that in fact this is a collection of maps

$$\psi_j : \Lambda_{(j)}(\Sigma_0) \to \Lambda_{(j)}(\Sigma_1),$$

using the normalised gradings as above. If $\Sigma_0 = \Sigma_1$ then one finds that the Alexander polynomial of the closed manifold $\overline{W}$ has co-efficients

$$a_j = (-1)^j \mathrm{Tr}(\psi_j).$$

Next consider a "nonabelian" version. Define a moduli space $\mathcal{N}_W$ of projectively flat connections, as before, which has a restriction map

$$\mathcal{N}_W \to \mathcal{N}(\Sigma_0) \times \mathcal{N}(\Sigma_1).$$

The image defines a homology class in the product which can be regarded as a map

$$\Psi(W) : H_*(\mathcal{N}(\Sigma_0)) \to H_*(\mathcal{N}(\Sigma_1)).$$

If $\Sigma_0 = \Sigma_1$ the Casson invariant $C(\overline{W})$ is a (graded) trace $\mathrm{Tr}(\Psi(W))$. Now one can prove an enhanced version of (17) which identifies the topological field theory Ψ with a sum of copies of the theories ψ_j. The advantage of this is that the functorial properties mean that it suffices to check on "elementary cobordisms".

There is a much deeper and more sophisticated "Jones-Witten" 2+1-dimensional TQFT. This assigns to a surface Σ vector spaces like the $V_k(\Sigma)$ mentioned at the end of Section 2 (Verlinde Theory). The invariants of a closed 3-manifold M are formally functional integrals

$$\int_{\mathcal{A}} e^{2\pi i k CS(A)} \mathcal{D}A.$$

4 Dimension 4

Special features of 4-dimensional geometry.

Let V be an oriented 4-dimensional Euclidean vector space. We have

$$* : \Lambda^2(V^*) \to \Lambda^2(V^*)$$

with $*^2 = 1$ so a decomposition into eigenspaces $\Lambda^+ \oplus \Lambda^-$. For $\alpha = \alpha_+ + \alpha_- \in \Lambda^2$ we have

$$\alpha \wedge \alpha = (|\alpha_+|^2 - |\alpha_-|^2)\mathrm{vol}.$$

The action on $\Lambda^\pm$ defines a $2-1$ local isomorphism of Lie groups $SO(4) \to SO(3) \times SO(3)$. Now the group Spin(4) is $SU(2) \times SU(2)$. Let S^+, S^- be 2-dimensional complex vector spaces with Hermitian metrics and complex symplectic forms. Combining these structures yields anti-linear maps $J : S^\pm \to S^\pm$ making these into 1-dimensional quaterninic vector spaces. A spin structure on V can be defined to be a pair of such spaces $S^\pm$ together with an isomorphism

$$\gamma : V \to \mathrm{Hom}_J(S^-, S^+)$$

to the maps which commute with J, taking the metric and orientation to standard data on $\mathrm{Hom}_J(S^-, S^+)$. Explicitly, in standard bases for $S^\pm$ we can a standard basis for V to γ_i given by:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The composites $\gamma(v_1)\gamma^*(v_2)$ define an isomorphism

$$\Lambda^+ = \mathbf{su}(S^+),$$

where the RHS is the Lie algebra of the group of special unitary transformations of S^+. Thus we can define a quadratic map

$$q : S^+ \to \Lambda^+$$

with $q(\phi)$ equal to the trace-free part of $i\phi\phi^*$.

In this section we will write

$$D : \Gamma(S^-) \to \Gamma(S^+)$$

for one part of the Dirac operator and $D^* : \Gamma(S^+) \to \Gamma(S^-)$ for its adjoint (making up the other part of the total Dirac operator discussed in Section 1). So in co-ordinates

$$D = \sum \gamma_i \frac{\partial}{\partial x_i}.$$

Now consider a compatible complex structure on V, making a complex Hermitian vector space. Then the self-dual forms decompose as

$$\Lambda^+ = \Lambda^{0,2} \oplus \mathbf{R}\omega,$$

where ω is the imaginary part of the Hermitian form. The spin space decomposes as

$$S^+ = K^{1/2} \oplus \Lambda^{0,2} \otimes K^{1/2}$$

while $S^- = \Lambda^{0,1} \otimes K^{1/2}$. Here $K^{1/2}$ is a square root of the complex line $\Lambda^2_{\mathbf{C}} V^*$. The structure map γ is given by wedge product and contraction. The Dirac operator D is identified with $\overline{\partial} + \overline{\partial}^*$.

The space of compatible complex structures is a 2-sphere $S(\Lambda^+) = \mathbf{P}(S^+)$. *A 2-form α is anti-self-dual if and only if its $(0,2)$ part vanishes for all choices of complex structure.*

The instanton equation

The standard model contains a Lagrangian, a functional on the connection and fields. The contribution from the connection alone is

$$S(A) = \|F(A)\|^2_{L^2}.$$

Functional integrals like

$$\int_{\mathcal{A}} e^{-\beta S(A)} \mathcal{D}(A),$$

appear in the theory with a parameter β. When β is large one expects that the main contribution comes from the minima of $\|F(A)\|$.

For an $SU(r)$ bundle E over a compact oriented Riemannian 4-manifold M we have

$$8\pi^2 c_2(P) = \int_M \mathrm{Tr}(F \wedge F) = \|F_-\|^2 - \|F_+\|^2.$$

If $c_2 \geq 0$ connections with $F^+ = 0$ minimise $S(A)$. Such connections are called ASD Yang-Mills instantons. If M has a complex structure an instanton defines a holomorphic structure on E.

On $\mathbf{R}^4$ the instanton condition is $F_{0i} + F_{jk} = 0$ where i, j, k run over cyclic permutations of (123), or in other words

$$[\nabla_0, \nabla_i] + [\nabla_j, \nabla_k] = 0.$$

Taking $\mathbf{R}^4 = \mathbf{C}^2$, two of these three equations are equivalent to

$$[\nabla_0 + \sqrt{-1}\nabla_1, \nabla_2 + \sqrt{-1}\nabla_3] = 0,$$

which is the integrability condition for the holomorphic structure.

The instanton equation is in a sense "integrable", in that there are procedures for writing down the general local solution (but involving solving a "Riemann-Hilbert problem"). Many well-known integrable equations can be obtained as reductions of the instanton equation. We illustrate this by considering *Nahm's equations*. These arise if we take connection matrices which only depend on 1-parameter x_0, so we have

$$A = \sum_{i=1}^{3} T_i(x_0) dx_i.$$

Write $x_0 = s$: the equations are

$$\frac{dT_i}{ds} = [T_j, T_k]$$

for (ijk) cyclic. Then we have

$$\frac{d}{ds}(T_1 + \sqrt{-1}T_2) = [\sqrt{-1}T_3, T_1 + \sqrt{-1}T_2],$$

which implies that the spectrum of $T_1 + \sqrt{-1}T_2$ does not change with s. There is a family of such equations parametrised by S^2. In terms of a complex co-ordinate ζ (omitting the point at infinity) we set

$$A(\zeta, s) = (1 + \zeta^2)T_1 + i(1 - \zeta^2)T_2 - 2i\zeta T_3,$$

and one finds that

$$\frac{d}{ds}A(\zeta, s) = [A(\zeta, s), -iT_3 + \zeta(T_1 - iT_2)].$$

So the *spectral curve* defined by the equation (polynomial in (ζ, η)):

$$\det(A(\zeta, s) - \eta) = 0,$$

is independent of s. More invariantly, we get a compact curve C in the total space of tangent bundle of S^2. The solution to Nahm's equation can be recovered via holomorphic geometry on this curve.

Remark

On a product $M = Y \times \mathbf{R}$ there are two different notions for a connection $\mathbf{A}$ on a bundle $\underline{P}$:

1. The pull-back of a connection A over Y;
2. We have a lift of the translation action to P and $\mathbf{A}$ is invriant under this lift. Such connections correspond to pairs (A, ϕ) where A is a connection on a bundle $P \to Y$ and ϕ is a section of ad P.

This gives various reductions of the instanton equation in 4 dimensions. In dimension 3 the first corresponds to flat connections over 3-manifolds. The second corresponds to *monopoles*: solutions of the equation

$$*_3 F_A = d_A \phi.$$

We have $|F_A|^2 = d(\mathrm{Tr}(\phi F_A)$ so any solution over a compact 3-manifold is flat, but there are interesting solutions on non-compact 3-manifolds.

Going down to dimension 2 we get the equations over a Riemann surface with metric for a bundle E and section Φ of End$E$:

$$\overline{\partial}_A \Phi = 0 \quad , \quad *_2 F_A = [\Phi, \Phi^*].$$

A variant gives *Hitchin's equations* for a connection A and $\Phi \in \Omega^{1,0}(\mathrm{End}E)$ which are conformally invariant:

$$\overline{\partial}_A \Phi = 0 \quad , \quad F_A = [\Phi, \Phi^*].$$

Spinors and Weitzenbock formulae

We introduce a formula which will be central in what follows. Given a connection A on a bundle E over $\mathbf{R}^4$ (for the moment) we have a coupled Dirac operator

$$D_A : \Gamma(E \otimes S^-) \to \Gamma(E \otimes S^+),$$

and $D_A D_A^*$ mapping $\Gamma(E \otimes S^+)$ to itself. We also have a covariant derivative ∇_A on $\Gamma(E \otimes S^+)$ and an operator $\nabla_A^* \nabla_A$. The "Weitzenbock" formula is

$$D_A D_A^* = \nabla_A^* \nabla_A + F_A^+, \tag{18}$$

where $F^+ \in \Lambda^+ \otimes \mathrm{End}E$ acts on $E \otimes S^+$ using the isomorphism $\Lambda^+ = \mathbf{su}(S^+)$. On a general oriented Riemannian 4-manifold with spin structure the formula is

$$D_A D_A^* = \nabla_A^* \nabla_A + F_A^+ + \frac{R}{4},$$

where R is the scalar curvature. The point is that for an instanton A the curvature from the bundle does not enter.

Connections on sub-bundles

Let E_1 be a vector bundle with connection and $\mathcal{H} \subset E_1$ a sub-bundle provided with a projection $p : E_1 \to \mathcal{H}$. For example this occurs if E_1 has a metric, using orthogonal projection. Then there is an induced connection on $\mathcal{H}$ given by the formula

$$\nabla_{\mathcal{H}}(s) = p \circ (\nabla_E s).$$

Now suppose that the base manifold is complex and that we have three holomorphic bundles with holomorphic bundle maps

$$E_0 \xrightarrow{\alpha} E_1 \xrightarrow{\beta} E_2,$$

such that $\beta \circ \alpha = 0$. Suppose that α is injective and β is surjective which implies that the cohomology $H^1 = \mathrm{Ker}\beta/\mathrm{Im}\alpha$ forms another holomorphic bundle. If E_i have Hermitian metrics we can identify H^1, as a C^∞ bundle, with

$$\mathcal{H} = \ker(\alpha^* + \beta).$$

which is a sub-bundle of E_1. The simple fact that we need is that the induced connection on $\mathcal{H}$ is compatible with the holomorphic structure it inherits from the identification with H^1.

Instantons on 4-tori: the Fourier transform

Let $T = \mathbf{R}^4/\Lambda$ be a Riemannian 4-torus. There is a dual torus

$$\hat{T} = \mathrm{Hom}(\Lambda, S^1).$$

A point $\xi \in \hat{T}$ defines a flat unitary line bundle $L_\xi \to T$.

Now let A be an irreducible instanton connection on a $U(r)$ bundle $E \to T$. For each ξ we have Dirac operators $D_{A,\xi}, D^*_{A,\xi}$ on spinors coupled to $E \otimes L_\xi$. The Weitzenbock formula (18) implies that $\ker D^*_{A,\xi} = 0$. (More precisely, the condition on A that we need is that it does not split off any flat summand.) This implies that $\ker D_{A,\xi}$ has constant dimension and these kernels form a bundle $\hat{E}$ over $\hat{T}$. Locally in $\hat{T}$ we can regard L_ξ as the trivial bundle with a deformed connection, so the kernels can be regarded as subspaces of a fixed vector space. Then we get an connection $\hat{A}$ on $\hat{E}$ induced by L^2 orthogonal projection. (The infinite dimensionality does not cause real problems.) Then we have:

The connection $\hat{A}$ is an instanton over $\hat{T}$.

We can see this without calculation. Given a complex structure on $\mathbf{R}^4$, we regard E as a holomorphic bundle over the complex torus T. Then we have an analogous holomorphic construction defining a bundle over $\hat{T}$: for each ξ we take the cohomology $H^1(E \otimes L_\xi)$. This is clearly a holomorphic bundle over $\hat{T}$ and it can be identified with the Dirac operator kernel bundle $\hat{E}$ via the equality $D = \overline{\partial} + \overline{\partial}^*$ and Hodge Theory. The simple fact above shows that the connection on $\hat{E}$ is compatible with the holomorphic structure, so in particular $F^{0,2} = 0$. But this holds for *all* complex structures so the connection has $F^{+} = 0$.

The dual of $\hat{T}$ is T, so we can repeat the construction starting with $\hat{A}$ to get another instanton over T. In fact this agrees with $\iota^* A$ where $\iota : T \to T$ is the map $x \mapsto -x$. There are obvious analogies with the Fourier transform. We may say more about the proof later.

The Atiyah-Singer Index Theorem (or Riemann-Roch) show that this transform interchanges the rank and second Chern class. One application is to show that there is no instanton over T with $c_2 = 1$.

The ADHM construction

This gives a (reasonably) explicit description of all instantons over S^4. The equations are conformally invariant so we work over $\mathbf{R}^4$. The construction requires the following data.

- Complex Hermitian vector spaces U, E_∞ of dimensions k, r respectively.
- Self-adjoint maps $T_i : U \to U$ (i=0,...,3).
- A linear map $P : E_\infty \to U \otimes S^+$

For each point $x \in \mathbf{R}^4$ we define

$$R_x : U \otimes S^- \oplus E_\infty \to U \otimes S^+$$

by

$$\sum (T_i - x_i) \otimes \gamma_i \oplus P.$$

We need the data to satisfy

- A non-degeneracy condition: R_x is surjective for each $x \in \mathbf{R}^4$.
- The ADHM equations:

$$[T_0, T_i] + [T_j, T_k] + q_U(P)_i = 0. \tag{19}$$

Here $q_U(P) \in \mathrm{End}(U) \otimes \Lambda^+$ is formed using the quadratic map q above on the spinors, the metric on E_∞, and multiplication $U \otimes \overline{U}^* \to \mathrm{End}(U)$. The subscript i denotes the component in our standard basis of Λ^+. (Of course we could write this more invariantly.)

The non-degeneracy condition means that we can form a bundle E over $\mathbf{R}^4$ with fibres $E_x = \mathrm{Ker}\ R_x$. This is a sub-bundle of a trivial Hermitian bundle so we get an induced connection. One checks that the construction extends smoothly over the point at infinity with fibre E_∞. The statement is that this connection is an instanton and all arise this way (for matrix data which is unique up to natural equivalence). The bundle E has rank r and $c_2 = k$.

As for tori, we understand this construction by choosing a complex structure $\mathbf{R}^4 = \mathbf{C}^2$. Then we set

$$\tau_1 = T_0 + \sqrt{-1}T_1, \tau_2 = T_2 + \sqrt{-1}T_3.$$

Under the decomposition of S^+ , and taking the adjoint of one component, the map P gives a pair $\pi : U \to E_\infty, \sigma : E_\infty \to U$. For $z = (z_1, z_2) \in \mathbf{C}^2$ we have maps

$$\alpha_z = ((\tau_1 - z_1 1)\ (\tau_2 - z_2 1)\ \pi) : U \to U \oplus U \oplus E_\infty$$

$$\beta_z = \begin{pmatrix} (-(\tau_2 - z_2 1) \\ (\tau_1 - z_1 1) \\ \sigma \end{pmatrix} U \oplus U \oplus E_\infty \to U$$

with $R_z = \alpha_z^* + \beta_z$ and two of the three ADHM equations say that this is a complex : $\beta_z \alpha_z = 0$, i.e.

$$[\tau_1, \tau_2] + \sigma\pi = 0.$$

The remaining equation is

$$[\tau_1, \tau_1^*] + [\tau_2, \tau_2^*] - \pi^*\pi + \sigma\sigma^* = 0. \tag{20}$$

The same argument as before shows that the connection has $F^{0,2} = 0$ and since this is true for all complex structures it is an instanton.

The ADHM result can be proved via a discussion of holomorphic bundles on $\mathbf{CP}^2$. There is a smooth map $\mathbf{CP}^2 \to S^4$ collapsing the line at infinity to a point so our instanton defines a holomorphic bundle over $\mathbf{CP}^2$ which we still call E. This is clearly trivial over the line at infinity. The projective version of the construction above is a "monad"

$$U(-1) \xrightarrow{\alpha} V \xrightarrow{\beta} U(1),$$

with $\beta\ \alpha = 0$. Here the notation $U(-1)$ means U tensored with the dual of the hyperplane line bundle $\mathcal{O}(1)$ and V is regarded as the trivial bundle with fibre V. There are three steps in this approach to proving the ADHM result.

1. Any bundle over $\mathbf{CP}^2$ which is trivial over some line arises as the cohomology of a monad.
2. The monad can be chosen to satisfy the remaining equation (20).
3. The connection we get agrees with the original one.

Step 1

Let $E \to \mathbf{P}^2$ be trivial over some line and set $F = E(-1)$. Fix a point in $\mathbf{CP}^2$, say the origin in $\mathbf{C}^2$. Over $\mathbf{C}^2$ we have a radial vector field $z_1\partial_1 + z_2\partial_2$ which extends to a section of the bundle $Q = T\mathbf{P}^2(-1)$ vanishing only at the chosen point. Now form the Koszul complex

$$\Lambda^2 Q^* \to Q^* \to \mathcal{O}.$$

which gives a resolution of the skyscraper sheaf at our point. One finds that $\Lambda^2 Q^* = \mathcal{O}(-1)$. Tensor with F so we have

$$F(-1) \to F \otimes Q^* \to F.$$

Such a complex of sheaf maps gives a pair of spectral sequences converging to the same limit (the hypercohomology). From one sequence we get the fibre F_0 of F at our point. From the other we get an E_1 term

$$H^q(F(-1)) \quad , \quad H^q(F \otimes Q^*), \quad H^q(F).$$

Standard arguments, using the fact that E is trivial on some line, show that $H^0(F(-1)), H^0(F), H^2(F(-1)), H^2(F)$ all vanish and $H^1(F(-1)) = H^1(F)$. We deduce from the spectral sequence that $H^q(F \otimes Q^*) = 0$ for $q \neq 1$. The E_1 term of the sequence says that the fibre F_0 is the cohomology of a complex

$$U \to V \to U,$$

with $U = H^1(F)$ and $V = H^1(F \otimes Q^*)$. Letting the chosen point vary over $\mathbf{CP}^2$ we get our monad. (Another way of expressing this is through a resolution of the diagonal in $\mathbf{CP}^2 \times \mathbf{CP}^2$: the resulting spectral sequence is the *Beilinson spectral sequence.*)

Step 2 It is straightforward to check that any monad for a bundle over $\mathbf{P}^2$ trivial on the line at infinity can be put in our standard form as above, built from data $(\tau_1, \tau_2, \pi, \sigma)$. The group $GL(U)$ of automorphisms of U act on the data and the moment map for the action of the unitary subgroup is the expression on the left hand side of (20). Also the nondegeneracy condition gives stability. Then the required result follows from the identification of complex and symplectic quotients.

Step 3 Start with an instanton connection A on a bundle E. Regarded as a holomorphic bundle over $\mathbf{CP}^2$, we now get another instanton connection A' on E from the monad construction and we want to show that $A = A'$. More precisely, the monad construction gives a connection A' for any choice of trivialisation of E over the line at infinity and we are allowed to vary this trivialisation.

To explain the argument we regard A' as a connection on a bundle E'. So we have a map $\phi \in \mathrm{Hom}(E, E')$ which gives an isomorphism of holomorphic structures. That is $\overline{\partial}_{A,A'}\phi = 0$, where $\overline{\partial}_{A,A'}$ is the operator defined by the induced connection on $\mathrm{Hom}(E, E')$. Then, regarding ϕ as a bundle valued spinor, the Weitzenbock formula (18) gives

$$\nabla^* \nabla \phi = 0,$$

where ∇ is the covariant derivative defined by A, A'. Set $\tau = \mathrm{Tr}(\phi^* \phi)$. It follows that

$$\Delta \tau = -|\nabla \phi|^2.$$

We can suppose that the determinant of $\phi^*\phi$ is 1, so $\tau \geq r$ with equality if and only if $\phi^*\phi$ is the identity. By varying the trivialisation we can suppose that $\tau \to r$ at infinity. Then the maximum principle implies that τ is identically r and $\nabla\phi = 0$. This means that ϕ is an isomorphism of bundles with connections.

There are many approaches to this ADHM construction. One (the original one) goes through *twistor theory.* We regard S^4 as the quaternionic projective line $\mathbf{PH}^1$. Then the identification $\mathbf{H}^2 = \mathbf{C}^4$ gives a fibration $\pi : \mathbf{CP}^3 \to S^4$ whose fibres are complex projective lines. If A is an instanton connection on $E \to S^4$ the pull-back $\pi^*(A)$ defines a holomorphic structure on the bundle $\pi^*(E) \to \mathbf{CP}^3$ which is holomorphically trivial on the fibres of π. Conversely given a holomorphic bundle over $\mathbf{P}^3$ trivial on these lines one can recover an instanton on S^4. This "Ward construction" is local: it applies to instantons on an open set $U \subset S^4$ and holomorphic bundles over $\pi^{-1}(U)$. The Riemann-Hilbert problem we mentioned above is to find a traivialisation of a holomorphic bundle over $\mathbf{CP}^1$ given a transition function over a neighbourhood of the equator.

Applications to 4-manifold (differential) topology

The dimension of the moduli space of $SU(2)$-instantons over S^4 with $c_2 = k$ is $8k-3$ (which agrees with the naive count from the ADHM data). In general the deformation theory of an instanton on a bundle E over a compact oriented 4-manifold M is governed by a deformation complex

$$\Omega^0(\mathrm{ad}E) \stackrel{d_A}{\to} \Omega^1(\mathrm{ad}E) \stackrel{d_A^+}{\to} \Omega^+(\mathrm{ad}E).$$

The "virtual dimension" of this moduli space is the Euler characteristic of this complex which is given by the Atiyah-Singer index theorem. For an SU(2) bundle with $c_2 = k$ it is

$$\mathrm{ind}(A) = 8k - 3(1 - b_1 + b^+),$$

where b_1 is the first Betti number and b^+ is the dimension of a maximal positive subspace for the cup product form on $H^2(M)$. If we use $U(2)$-bundles and connections which are projectively ASD and with fixed determinant we get dimension

$$ind(A) = 2(4c_2 - c_1^2) - 3(1 - b_1 + b^+).$$

In fact for a generic choice of Riemannian metric we do get moduli spaces of these dimensions, away from reducible connections.

The moduli spaces of instantons are generally not compact, but if the dimension is small and under certain other conditions they will be. Let us fix attention on a special case when M is a $K3$ surface and we choose a $U(2)$ bundle E with $4c_2 - c_1^2 = 6$. Since $b_1 = 0, b^+ = 3$ we have dimension 0 and for generic metrics our moduli space is a finite set. Since c_1 is not zero modulo 2 we avoid problems with reducible connections. We get an invariant of the pair (M, E) by counting the points in the moduli space. For maximal simplicity we ignore signs and get an invariant $n(M, E)$ in $\mathbf{Z}/2$.

We can calculate $n(M, E)$ using complex geometry. If M has a complex structure an extension of the Narasimhan-Seshadri theorem identifies instantons with stable holomorphic bundles. Suppose that E_0, E_1 are two stable holomorphic bundles of the given topological type. The Riemann-Roch Theorem tells us that the holomorphic Euler characteristic of the bundle $\mathrm{Hom}(E_0, E_1)$ is 2. By Serre duality (since the canonical bundle is trivial) there is a holomorphic map from E_0 to E_1 or from E_1 to E_0 (or both). An extension of the result we discussed in Section 2 shows that any holomorphic map between these stable bundles is an isomorphism. So there is at most one point in our moduli space. To see that there is at least one point, take a complex $K3$ surface which is a double branched cover $\pi : M \to \mathbf{CP}^2$. Then the pull back $\pi^*(T\mathbf{CP}^2)$ is stable and has the correct topological type. So $n(M, E) = 1$.

The non-triviality of this invariant has important consequences. Suppose that M were a connected sum $M_1 \sharp M_2$. To avoid problems with reducibles we want to assume that $c_1(E)$ does not vanish mod 2 in each piece and that $b^+(M_i) > 0$. Take a sequence of metrics on M with the neck of the connected sum shrinking to zero size. One can show that if there are instantons on M

for each of these metrics then in the limit we get instantons A_1, A_2 say on M_1, M_2. When the neck is small the connections on M can be thought of as being obtained by gluing: schematically $A_1 \sharp A_2$. The dimension formula gives

$$\mathrm{ind}(A_1 \sharp A_2) = \mathrm{ind}(A_1) + \mathrm{ind}(A_2) + 3.$$

Since $\mathrm{ind}(A_1 \sharp A_2) = 0$ this shows that one of A_i lives in a negative-dimensional moduli space, which gives a contradiction. We deduce that M cannot be decomposed as a connected sum of this kind.

The Seiberg-Witten equations

Now we consider a compact, oriented, Riemannian 4-manifold M with spin structure. Let $L \to M$ be a unitary line bundle. The Seiberg-Witten equations are for a pair (A, ϕ) where A is a connection on L and ϕ is a section of $S^+ \otimes L$. The equations are:

$$D_A^* \phi = 0 \quad , \quad F_A^+ = i q_L(\phi).$$

Here q_L is the quadratic map from $S^+ \otimes L$ to Λ^+ defined using q and the metric on L. The remarkable feature of these equations is that the moduli space is always compact. The Weitzenbock formula gives

$$0 = D_A D_A^* \phi = \nabla_A^* \nabla_A \phi + F_A^+ \phi + \frac{R}{4} \phi.$$

Taking the L^2 inner product with ϕ we get

$$\int_M |\nabla_A \phi|^2 + (F_A^+ \phi, \phi) + \frac{R}{4} |\phi|^2.$$

Now

$$(F_A^+ \phi, \phi) = (i q_L(\phi) \phi, \phi),$$

and (with the right choice of sign conventions) this is $|\phi|^4$. So if $R/4 \geq -C$ say we have

$$\int_M |\nabla_A \phi|^2 + |\phi|^4 \leq C \int_M |\phi|^2.$$

Using Cauchy-Schwartz we get

$$\int_M |\phi^4| \leq C \mathrm{Vol}(M).$$

Standard elliptic estimates then give bounds on all derivatives of the connection and ϕ, in a suitable gauge.

The "virtual dimension" of the moduli space of solutions modulo gauge equivalence is

$$\mathrm{ind}_{SW}(L) = c_1(L)^2 - (1 - b_1 + b^+) + \frac{1}{4}(b^- - b^+).$$

We assume L chosen so that this number is 0. To avoid problems with reducible solutions (i.e. those with $\phi = 0$) we assume that $b^+(M) > 1$. Then we get a Seiberg-Witten invariant $\mathrm{SW}(M, c)$, where $c = c_1(L) \in H^2(M)$, by "counting" the solutions. These can be used in the same manner as the instanton invariants to get differential-topological information about 4-manifolds and in fact give generally stronger results.

If M is a Kähler surface with spin structure there is a decomposition $S^+ = K^{1/2} \oplus \Lambda^{0,2} \otimes K^{1/2}$ compatible with the connection on the spin bundle. We take $L = K^{1/2}$ so $S^+ \otimes L = \mathbf{C} \oplus \Lambda^{0,1}$. This satisfies $\mathrm{ind}_{SW}(L) = 0$ (which is equivalent to *Noether's formula*). Supppose that in fact the metric has constant negative scalar curvature. Then there is solution of the Seiberg-Witten equations given by taking ϕ to be a suitable multiple of the section 1 of the $\mathbf{C}$ factor in $S^+ \otimes L$ and the standard connection on $K^{1/2}$. It is not hard to show that this is the only solution, so the Seiberg-Witten invariant is 1. This discussion generalises to all Kähler surfaces and in fact all symplectic 4-manifolds (assuming $b^+ > 1$). On the other hand we have vanishing results for connected sums, similar to the instanton case.

As another kind of application, one can give a simple proof that if M is a simply connected spin 4-manifold with *negative definite intersection form* then in fact $H^2(M) = 0$, i.e. M is a homotopy 4-sphere. For this we consider the trivial line bundle L. The moduli space has virtual dimension $B/4 - 1$ where $B = \text{dim} H^2(M)$. It is a relatively easy fact from topology (or the theory of quadratic forms) that B is divisible by 8 so we write $B = 8r$. If $B > 0$ the moduli space has a singular point corresponding to the "trivial" reducible solution (the flat connection A and $\phi = 0$). A neighbourhood of this point is modelled on a cone over $\mathbf{CP}^{r-1}$. Cutting off the cone point, we get a compact manifold $\mathcal{M}$ with boundary $\mathbf{CP}^{r-1}$. But there is a class $b \in H^2(\mathcal{M})$ (defined in an analogous way to the class $b \in H^4$ in the case of moduli spaces of rank 2 bundles) which restricts to the standard generator of $H^2(\mathbf{CP}^{r-1})$ and this gives a contradiction. So we see that $B = 0$.

Note that a simply connected compact 4-manifold is spin if and only if it has *even* intersection form. Given n divisible by 8, there is a *Siegel formula* for a weighted sum over isomorphism classes of negative definite, even unimodular quadratic forms of rank n in terms of Bernoulli numbers:

$$\sum_{\Lambda} \frac{1}{\text{Aut}(\Lambda)} = |B_{n/2}| \prod_{i \leq j < n/2} \frac{|B_{2j}|}{4j}.$$

Since there at least two automorphisms, this gives a lower bound on the count of isomorphism classes. For $n = 32$ there are at least 80 million. None of these forms can appear as the intersection form of a (simply connected) smooth 4-manifold, but they are all realised by *topological* 4-manifolds. In general the results obtained from the instanton and Seiberg-Witten equations do not apply to topological 4-manifolds where the situation is completely different.

Some references

Griffiths and Harris is a good general reference, in particular for connections on holomorphic bundles. (But beware that their notation is different so for example their formula for the curvature in terms of a connection matrix is $dA - A \wedge A$.)

Much of the material in the course is covered in **Donaldson and Kronheimer: The Geometry of 4-manifolds, OUP 1990** (with an emphasis on the instanton invariants of 4-manifolds).

A reference for results of Narasihman-Seshadri type is **Lubke and Teleman: The Kobayashi-Hitchin correspondence. World Scientific 1995**

For the homology of moduli spaces of stable bundles, and much other relevant material see **Atiyah and Bott: The Yang-Mills equations over Riemann surfaces, Phil Trans. Roy Soc. 1982**

The formulae for volumes of moduli spaces go back to **Witten: On quantum gauge theories in 2 dimensions, Commun Math Phys 1991** and **Thaddeus: Conformal Field Theory and cohomology of moduli spaces of stable bundles, Jour. Diff. Geom. 1992**. The approach we discuss above is outlined in the last part of **Donaldson: Gluing techniques in the cohomology of moduli spaces. In: Topological methods in modern mathematics, Publish and Perish 1993**

The description of the moduli space of bundles over a curve of genus 2 is in **Newstead: Stable bundles of rank 2 and odd degree over a curve of genus 2. Topology 1968** and **Narasimhan and Ramanan: Annals of Math. 1969**.

Casson's original construction and proof of the surgery formula is in **Akbulut and McCarthy: Casson's invariant for oriented homology 3-spheres, Princeton UP 1990**. The TQFT approach we outline is developed in more detail in **Donaldson: Topological Field Theories and formulae of Casson and Meng-Taubes: In Geometry and Topology monographs, Vol 2 1999**

The original reference for the Seiberg-Witten theory is **Witten: Monopoles and 4-manifolds: Math. Reseach Letters 1994** There is an early survey article **Donaldson: The Seiberg-Witten equations and 4-manifold topology, Bull. AMS 1996** and various book treatments such as **Morgan: The Seiberg-Witten equations and applications to the topology of 4-manifolds, Princeton UP 1996**

The spectral curve construction for Nahm's equations is in **Hitchin: On the construction of monopoles, Commun. Math. Phys 1983**

The Belinson spectral sequence is in **Okonek, Schneider and Spindler: Vector bundles on complex projective spaces, Birhauser 1980**

A good reference for the original treatment of the ADHM construction, via twistor space, is **Atiyah: The geometry of Yang-Mills fields, Fermi Lectures, Pisa 1979** (May be hard to find.)